数理解析研究所講究録
第 1949 巻 2015 年 73-80

Dominating mad families in Baire space

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ABSTRACT. In this note we consider a Marczewski like nonmeasurable sets (with respect to trees) which forms m.a.d. family in Baire space. Here we show that under assumption that $\omega_1 = \mathfrak{b}$ there is a m.a.d. family in Baire space which is not s-measurable (here we can replace s-nonmeasurable by l-nonmeasurable or m-nonmeasurable). Moreover it is relatively consistent with ZFC theory that $\omega_1 < \mathfrak{d} \leq \mathfrak{c}$ and there is m.a.d. family in Baire space which is not measurable with respect to family of all complete Laver trees in ω^ω.

1. Definitions

We adopt the standard set theoretic notation ω stands for first infinite ordinal, $\mathfrak{c}$ is denoted as size of all reals, for any set X, $|X|$ is size of X, $P(X)$ is power set of X, $[X]^\kappa$ is denoted as set of all subsets of X of the cardinality κ, $X^{<\kappa}$ denotes the set of all sequences in X with lenght less than κ. We say that for $T \subseteq \omega^{<\omega}$ the partial order $(T, \subseteq)$ is tree if for any $\tau \in T$ and $n \in dom(\tau)$ we have $\tau \restriction n \in T$. By the set

$$[T] = \{x \in \omega^\omega : (\forall n \in \omega) x \restriction n \in T\}$$

we denote envelope of T.

Now we turn into notion of measurability with respect to a fixed families of trees on the Baire space.

Edward Marczewski [**6**] introduced notion of s measurability and s_0-ideal notion.

DEFINITION 1.1 (Marczewski ideal s_0). *Let X be any fixed uncountable Polish space. Then we say that $A \in \mathcal{P}(X)$ is in s_0 iff*

$$(\forall P \in Perf(X))(\exists Q \in Perf(X))\ Q \subseteq P \wedge Q \cap A = \emptyset.$$

Of course every perfect set is an envelope of some perfect tree and the above definition can be formulated in tree terms.

DEFINITION 1.2 (s measurable set). *Let X be any fixed uncountable Polish space. Then we say that $A \in \mathcal{P}(X)$ is* s-**measurable** *iff*

$$(\forall P \in \mathrm{Perf}(X))(\exists Q \in \mathrm{Perf}(X))\ Q \subseteq P \wedge (Q \subseteq P \vee Q \cap A = \emptyset).$$

Here let us recall the notion of the Laver tree. Then we say that tree $T \subseteq \omega^{<\omega}$ is called a **Laver tree** with the stem $s \in T$ if

- for any $t \in T$ we have $t \subset s \vee s \subseteq t$,
- for every node $t \in T$ if $s \subseteq t$ then t is infinitely spliting i.e. $\{n \in \omega : t^\frown n \in T\}$ is an infinite.

Miller tree $T \subseteq \omega^{<\omega}$ with stem $s \in T$ is defined in the same manner but the second condition is replaced by the following

$$(\forall t \in T)(s \subseteq t) \longrightarrow (\exists r \in T)(t \subseteq r) \wedge (\{n \in \omega : r^\frown n \in T\} \in [\omega]^\omega).$$

The we can recall a similar definition of the ideal l_0 to the previous one. The set of all Laver trees is denoted by the LaverTrees.

Definition 1.3 (ideal l_0). *We say that $A \in \mathcal{P}(\omega^\omega)$ is in l_0 iff*

$$(\forall T \in \text{LaverTrees})(\exists Q \in \text{LaverTrees})\ Q \subseteq T \wedge [Q] \cap A = \emptyset.$$

Definition 1.4 (l measurable set). *We say that $A \in \mathcal{P}(\omega^\omega)$ is l-* **measurable** *iff for every Laver tree $T \in$ LaverTrees there is a Laver tree $S \in$ LaverTrees such that*

$$(S \subseteq T \wedge [S] \subseteq A) \vee (S \subseteq T \wedge [S] \cap A = \emptyset).$$

We say that tree $T \subseteq \omega^{<\omega}$ is called a **complete Laver tree** iff every node $t \in T$ is infinitely spliting.

Then once again we can recall a similar definition of the ideal cl_0 to the previous one. The set of all complete Laver trees is denoted by the cLaver.

Definition 1.5 (ideal cl_0). *We say that $A \in \mathcal{P}(\omega^\omega)$ is in cl_0 iff*

$$(\forall T \in \text{cLaver})(\exists Q \in \text{cLaver})\ Q \subseteq T \wedge [Q] \cap A = \emptyset.$$

Definition 1.6 (cl measurable set). *We say that $A \in \mathcal{P}(\omega^\omega)$ is* cl-**measurable** *iff for every complete Laver tree $T \in$ cLaver there is a complete Laver tree $S \in$ cLaver such that*

$$(S \subseteq T \wedge [S] \subseteq A) \vee (S \subseteq T \wedge [S] \cap A = \emptyset).$$

As above using notion of Miller tree we can define m-measurability and notion of m_0-ideal.

Next we recall the notion of almost disjoint family in Baire space.

Definition 1.7. *We say that family $\mathcal{A} \subseteq \omega^\omega$ is* **a.d.** *family in Baire space if*

$$(\forall a, b \in \mathcal{A})\ a \neq b \longrightarrow a \cap b \text{ is finite.}$$

If this family is maximal with respect to inclusion in Baire space then $\mathcal{A}$ is called **m.a.d.** *family in ω^ω.*

Now let us reacall cardinal $\mathfrak{d}$ as smallest size of dominating family in ω^ω i.e.

$$\mathfrak{d} = min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge (\forall g \in \omega^\omega)(\exists f \in \mathcal{F})(\forall^\infty n)\ g(n) < f(n)\}.$$

2. Dominating MAD families in Baire space and nonmeasurability with respect to ideals defined by trees

It is well known that every **a.d.** family is meager subset of the Baire space. It is natural to ask whether one can prove in ZFC the existence a **m.a.d.** families that are either s-measurable or s-nonmeasurable. One can find a consistency example of **m.a.d.** family $\mathcal{A}$ of cardinality smaller than $\mathfrak{c}$ (see [**5**], for example) by construction of Cohen indestructible m.a.d. family. One can find more about tree-like forcing indestructible m.a.d. familes in [**2**]. It is well known that $\mathrm{non}(s_0) = \mathfrak{c}$ (for other coefficients see [**1, 3, 4, 7**]) where $non(I)$ is smallest size of subset in ω^ω which does not belong to σ-ideal $I \subset P(\omega^\omega)$. It is well known that there exists a perfect **a.d.** family and therefore not all **m.a.d.** families are in s_0.

THEOREM 2.1. *There exists a s-nonmeasurable m.a.d. family in Baire space. Moreover, theorem remains true if we replace s-nonmeasurability by l, m or cl-nonmeasurability.*

PROOF. We show this theorem for s-nonmeasurability, for the other notion mentioned in above theorem the proof runs in analogous way. Let $T \subseteq \omega < \omega$ a perfect tree such that $[T]$ is a.d. in ω^ω. Let us enumerate $Perf(T) = \{T_\alpha : \alpha < \mathfrak{c}\}$ a family of all perfect subsets of T. By transfinite reccursion let us define

$$\{(a_\alpha, d_\alpha, x_\alpha) \in [T]^2 \times \omega^\omega : \alpha < \mathfrak{c}\}$$

such thatfor any $\alpha < \mathfrak{c}$ we have:

(1) $\{a_\xi : \xi < \alpha\} \cap \{d_\xi : \xi < \alpha\} = \emptyset$,
(2) $\{a_\xi : \xi < \alpha\} \cup \{x_\xi : \xi < \alpha\}$ is a.d.,
(3) $\forall^\infty n \; x_\alpha(n) = d_\alpha(n)$.

Now assume that we are in α-th step construction and we have required sequence

$$\{(a_\xi, d_\xi, x_\xi) \in [T]^2 \times \omega^\omega : \xi < \alpha\}$$

which have size at most $\omega|\alpha| < \mathfrak{c}$ then we can choose in $[T_\alpha]$ (of size $\mathfrak{c}$) $a_\alpha, d_\alpha \in [T_\alpha]$ which fulfills the first condition. Then choose any $x_\alpha \in \omega^\omega$ different than d_α but $(\forall^\infty n) d_\alpha(n) = x_\alpha(n)$ then $x \in \omega^\omega \setminus [T]$ and

$$\{a_\xi : \xi < \alpha\} \cup \{x_\xi : \xi < \alpha\}$$

forms an a.d. family in ω^ω. Then α-th step construction is completed. By transfinite induction theorem we have required sequence of the length $\mathfrak{c}$. Now set $A_0 = \{a_\alpha : \alpha < \mathfrak{c}\} \cup \{x_\alpha : \alpha < \mathfrak{c}\}$ and let us extend it to any maximal a.d. family A. It is easy to chect that A is required s-nonmeasurable m.a.d. family in the Baire space ω^ω. $\square$

Here we have obtained a consistency result but the above statement remains true in every model of ZFC theory whenever $\mathfrak{d} = \omega_1$.

THEOREM 2.2. *It $\mathfrak{d} = \omega_1$ then there exists a m.a.d. family of functions $\mathcal{A} \subseteq \omega^\omega$ such that $\mathcal{A}$ is not s-measurable and there is an dominating subfamily $\mathcal{A}' \in [\mathcal{A}]^{\leq \mathfrak{d}}$ in Baire space ω^ω. Moreover, the words not s-measurable can be replaced by not l, m and cl-measurable.*

PROOF. Now by assumption there is a dominating family $\mathcal{A} \subseteq \omega^\omega$ of size ω_1. Then we can show that we can find an a.d. dominating family of size ω_1. To do let us enumerate $\mathcal{A} = \{f_\xi : \xi < \omega_1\}$ and assume that we are in α-setp of construction with a.d. family $\mathcal{D}_\alpha = \{g_\xi : \xi < \alpha\}$ such that for any $\xi < \alpha$ we have $f_\xi \leq g_\xi$. Now let $\{h_n : n \in \omega\}$ be enumeration of $\mathcal{D}_\alpha$ then for any $n \in \omega$ let

$$g_\alpha(n) = \max\{f_\alpha(n), \max\{h_k(n) : k \leq n\}\} + 1.$$

Then the family $\mathcal{D} = \bigcup_{\alpha<\omega_1} \mathcal{F}_\alpha$ is as was required almost disjoint and dominating family of size equal to ω_1. Moreover, one can assume tha each member of $\mathcal{D}$ has even values only. Now let us fix a perfect tree S with the porperty that each member of S has odd values only.

Then we are ready to find a m.a.d. family $\mathcal{B}$ which is not s-measurable (in the perfect set $[S]$) and $\mathcal{D} \subseteq \mathcal{B}$.

Let us enumerate $Perf(S) = \{T_\alpha : \alpha < \mathfrak{c}\}$ a family of all perfect subsets of S. By transfinite reccursion let us define

$$\{(a_\alpha, d_\alpha, x_\alpha) \in [S]^2 \times \omega^\omega : \alpha < \mathfrak{c}\}$$

such that for any $\alpha < \mathfrak{c}$ we have:

(1) $a_\alpha, d_\alpha \in T_\alpha$,
(2) $\{a_\xi : \xi < \alpha\} \cap \{d_\xi : \xi < \alpha\} = \emptyset$,
(3) $\{a_\xi : \xi < \alpha\} \cup \{x_\xi : \xi < \alpha\}$ is a.d.,
(4) $\forall^\infty n\ x_\alpha(n) = d_\alpha(n)$ but $x_\alpha \neq d_\alpha$.

Now assume that we are in α-th step construction and we have required sequence

$$\{(a_\xi, d_\xi, x_\xi) \in [S]^2 \times \omega^\omega : \xi < \alpha\}$$

which have size at most $\omega|\alpha| < \mathfrak{c}$ then we can choose in $[T_\alpha]$ (of size $\mathfrak{c}$) $a_\alpha, d_\alpha \in [T_\alpha]$ which fulfills the first condition. Then choose any $x_\alpha \in \omega^\omega$ different than d_α but $(\forall^\infty n) d_\alpha(n) = x_\alpha(n)$ then $x_\alpha \in \omega^\omega \setminus [S]$ and

$$\{a_\xi : \xi \leq \alpha\} \cup \{x_\xi : \xi \leq \alpha\}$$

forms an a.d. family in ω^ω. Then α-th step construction is completed. By transfinite induction theorem we have required sequence of the length $\mathfrak{c}$. Now set $A_0 = \mathcal{D} \cup \{a_\alpha : \alpha < \mathfrak{c}\} \cup \{x_\alpha : \alpha < \mathfrak{c}\}$ and let us extend it to any maximal a.d. family A. It is easy to check that A is required s-nonmeasurable m.a.d. family in the Baire space ω^ω. □

In contrast of the previously proven result, we show the consistency for $\omega_1 < \mathfrak{d}$ and existing a dominating cl-nonmeasurable **m.a.d.** –family of size $\mathfrak{d}$.

THEOREM 2.3. *It is relatively consistent with ZFC theory that $\omega_1 < \mathfrak{c}$ and there exists a* **m.a.d.** *family of functions $\mathcal{A} \subseteq \omega^\omega$ such that $\mathcal{A}$ is not cl-measurable. Moreover, there is a dominating subfamily $\mathcal{A}' \in [\mathcal{A}]^{\mathfrak{d}}$ and $\omega_1 < \mathfrak{d} \leq \mathfrak{c}$.*

PROOF. Let us consider the ground model V of GCH. We first choose any complete Laver tree $T \subseteq \omega^{<\omega}$ in V such that $[T]$ forms an **a.d.** family. Now, let us define a forcing notion $(Q_T, \leq)$ as follows: $p = (x_p, s_p^g, s_p^b, \mathcal{F}_p) \in Q_T$ iff

- $x_p \in \omega^{<\omega}$ and
- $s_p^g, s_p^b \in [T]^{<\omega}$ are finite trees and
- $\mathcal{F}_p \in [\omega^\omega]^{<\omega}$,

The order is defined as follows: for every $p = (x_p, s_p^g, s_p^b, \mathcal{F}_p) \in Q_T$ and $q = (x_q, s_q^g, s_q^b, \mathcal{F}_q) \in Q_T$ we have $p \leq q$ iff

(1) $x_q \subset x_p \wedge s_q^g \subseteq s_p^g \wedge s_q^b \subseteq s_p^b \wedge \mathcal{F}_q \subseteq \mathcal{F}_p$,
(2) $(\forall t \in s_p^g)(\forall k) x_p(k) = t(k) \longrightarrow t \restriction_{k+1} \in s_q^g \wedge x_p \restriction_{k+1} \subseteq x_q$,
(3) $(\forall h \in \mathcal{F}_q)(\forall k) h(k) \geq x_p(k) \longrightarrow x_p \restriction_{k+1} \subseteq x_q$,
(4) $(\forall h \in \mathcal{F}_q)(\forall t \in s_p^b)(\forall k)\ h(k) = t(k) \longrightarrow t \restriction_{k+1} \in s_q^b$,
(5) $(\forall h \in \mathcal{F}_q)(\forall t \in s_p^g)(\forall k)\ h(k) = t(k) \longrightarrow t \restriction_{k+1} \in s_q^g$.

CLAIM 2.4. *Q_T is σ-centered (and so is c.c.c.) forcing notion.*

PROOF. Let $I = \{(x, s^g, s^b) : x \in \omega^{<\omega} \wedge s^g, s^b \in [T]^{<\omega}\}$. For every $v = (x, s^g, s^b) \in I$ the set $Q_v = \{p \in Q_T : (x_p, s_p^g, s_p^b) = (x, s^g, s^b)\}$ is a centered subset of Q_T, because for any $p, q \in Q_v$ the condition $r = (x, s^g, s^b, \mathcal{F}_p \cup \mathcal{F}_q)$ from Q_v is a common extension of p and q. Since I is countable Q_T is σ-centered and hence it satisfies *c.c.c.* . $\square$

Let $G \subseteq Q_T$ be a generic filter over V and in $V[G]$ let

$$x_G = \bigcup \{x_p : p \in G\},$$

$$S_G^g = \{t \in T : (\exists p \in G)(\exists s \in s_p^g)\ t \subseteq s\},$$

$$S_G^b = \{t \in T : (\exists p \in G)(\exists s \in s_p^b)\ t \subseteq s\}.$$

It follows that $x_G \in \omega^\omega$ because the sets $D_n = \{p \in Q_T : |x_p| \geq n\}$ for $n \in \omega$ are dense.

CLAIM 2.5. *$\emptyset \neq [S_G^g] \subseteq [T]$ and $\emptyset \neq [S_G^b] \subseteq [T]$,*

PROOF. Fix $n \in \omega$, condition $p \in G$ $s \in s_p^g$ then the set $D_{s,n} = \{r \in Q_T : (\exists t \in s_r^g) n \leq |t| \wedge s \subseteq t\}$ is dense in the poset Q_T under p. To see it, let $q \leq p$ be any forcing condition. Then $s_p^g \subseteq s_q^g$ of course. Then because tree T is a complete Laver tree then one can find a sequence $t \in T$ such that $s \subseteq t$, $n \leq |t|$, $t \cap x_q = s \cap x_q$ and for every $h \in \mathcal{F}_q$ $h \cap t = h \cap s$. Then the condition $r = (x_q, s_q^g \cup \{t\}, s_q^b, \mathcal{F}_q)$ is stronger than q and $r \in D_{n,s}$ what shows that $D_{n,s}$ is dense under p.

Now by the above paragraph we can define recursively the following two sequences $\{s_n : n \in \omega\}$ and $\{p_n : n \in \omega\}$ such that for every $n \in \omega$ we have

- $p_0 = p$ and $p_{n+1} \leq p_n$ and $p_n \in G$,
- $s_0 = s$, $s_n \in s^g_{p_n}$, $n \leq |s_n|$ and $s_n \subseteq s_{n+1}$.

Then $z = \bigcup\{s_n : n \in \omega\}$ is an element of $[S^g_G]$. Then $[S^g_G]$ is nonempty. It is easy to see that every element of $[S^g_G]$ belongs to $[T]$ by the definition of the set $[S^g_G]$. The proof for $\emptyset \neq [S^b_G] \subseteq [T]$ is the same. □

Let us denote by cLaver(T) the collection of all complete Laver subtrees of the tree T.

CLAIM 2.6. *For every $T_1 \in \mathrm{cLaver}(T) \cap V$ there is $z \in [S^b_G] \cap [T_1]$ such that $z \cap x_G$ and $\{m \in \omega : z(m) \neq x_G(m)\}$ are infinite sets,*

PROOF. Let us choose $p \in G$ and any ground model complete Laver subtree $T_1 \subseteq T$. Then we will find three sequences $\{p_n : n \in \omega\}$, $\{y_n : n \in \omega\}$ and $\{s_n : n \in \omega\}$ such that for every $n \in \omega$ we have:

- $p_0 = p$, $p_{n+1} \leq p_n$ and $p_{n+1} \in G$,
- $s_n \in s^b_{p_n}$ and $s_n \subseteq s_{n+1} \in T_1$,
- $y_n = x_{p_n}$,
- there is $m > n$ such that $y_{n+1}(m) = s_{n+1}(m)$,
- there is $m' > n$ such that $y_{n+1}(m') \neq s_{n+1}(m')$.

Assume that we have three finite sequences $\{p_k : k \leq n\}$, $\{y_k : k \leq n\}$ and $\{s_n : k \leq n\}$ such that for every $k < n$ we have:

- $p_{k+1} \leq p_k$ and $p_{k+1} \in G$,
- $s_k \in s^b_{p_k}$ and $s_k \subseteq s_{k+1} \in T_1$,
- $y_k = x_{p_k}$,
- there is $m > k$ such that $y_{k+1}(m) = s_{k+1}(m)$,
- there is $m' > k$ such that $y_{k+1}(m') \neq s_{k+1}(m')$.

Then in particular we have $p_n \in G$, $y_n = x_{p_n}$ and $s_n \in s^b_{p_n} \cap T_1$. Now let us denote by the symbols D and E the following sets:

$$\{r \in Q_T : n{+}1 < |x_r| \wedge (\exists s \in s^b_r \cap T_1)(\exists m > n{+}1) s_n \subseteq s \wedge s(m) = x_r(m)\},$$

and

$$\{r \in Q_T : n{+}1 < |x_r| \wedge (\exists s \in s^b_r \cap T_1)(\exists m > n{+}1) s_n \subseteq s \wedge s(m) \neq x_r(m)\}$$

respectively.

We show that D is dense set in Q_T under the condition p_n. To do, fix any forcing condition $q \in Q_T$ such that $q \leq p_n$. We know that $s_n \in s^b_q$ because $q \leq p_n$ and $s_n \in T_1$. Moreover T_1 is a complete Laver tree then $\{n \in \omega : s^\frown n \in T_1\}$ is infinite and the sets s^g_q and $\mathcal{F}_q$ are finite. Then there is $x \in T$ and $s \in T_1$ such that $x_q \subseteq x$, $s_n \subseteq s$, $x(m) = s(m)$ for a some $m > n{+}1$ and for every $h \in \mathcal{F}_q$ $x \cap h = x_q \cap h$, for every $t \in s^g_q$ $x \cap t = x_q \cap t$. Then $r = (x, s^g_q, s^b_q \cup \{s\}, \mathcal{F}_q)$ is a stronger forcing condition than q and belongs to the set D and then

D is dense under p_n. The subtree T_1 is from ground model then D belongs to ground model V. The similar argument shows that the set E is a dense in Q_T by replacing $x(m) = s(m)$ for a some $m > n+1$ by the $x(m) \neq s(m)$ for a some $m > n+1$ in the above paragraph and E is in the ground model V of course. Then $r \in D \cap E \cap G \neq \emptyset$ for a some r and one can find a condition $p_{n+1} \in G$ which is a stronger than p_n and r. Then there exists $s \in s^b_{p_{n+1}}$ such that $s_n \subseteq s \in T_1$ such that $x_{p_{n+1}}(m) = s(m)$ for a some $m > n+1$. Then let $s_{n+1} = s$ and $y_n = x_{p_{n+1}}$. Then by induction hypothesis the sequences $\{p_n : n \in \omega\}$, $\{s_n : n \in \omega\}$, $\{y_n : n \in \omega\}$ with the above conditions exists.

It is easy to see that $z = \bigcup\{s_n : n \in \omega\} \in S^b_G \cap [T_1]$ and $z \cap x_G$ is infinite and we have $x_G = \bigcup\{y_n : n \in \omega\} = \bigcup\{x_{p_n} : n \in \omega\}$. □

CLAIM 2.7. *For every* $T_1 \in \text{cLaver}(T) \cap V$ *we have* $[S^g_G] \cap [T_1] \neq \emptyset$.

PROOF. Proof is similar to the previous one. □

CLAIM 2.8. *The following familes* $\{x_G\} \cup [S^g_G] \cup (\omega^\omega \cap V)$ *and* $[S^b_G] \cup (\omega^\omega \cap V)$ *are almost disjoint.*

PROOF. By standard argument, the order conditions (3) and (5) guaranties that $x_G \cap h$ and $z \cap h$ for any $z \in [S^g_G]$ are finite, where $h \in \omega^\omega \cap V$ is an any old real. To see that for any $z \in [S^g_G]$ the intersection $x_G \cap z$ is finite, let $\{s_n : n \in \omega\}$ and $\{p_n : n \in \omega\}$ are sequences witnessing that $z \in S^g_G$. If for any $n \in \omega$ the intersection $s_n \cap x_{p_n}$ is empty then $z \cap x_G = \emptyset$ also. Then let assume that $n_0 \in \omega$ be a first positive integer such that intersection $x_{p_{n_0}} \cap s_{n_0}$ is nonempty. Let us choose an any integer n greater than n_0 such that there are no $s \in s^g_{p_{n_0}}$ such that $s_n \subset s$. Then by the point (2) of the definition of order between p_n and p_{n_0} we have $x_{p_n} \cap s_n \subseteq x_{p_{n_0}} \cap s_{n_0}$, (here $s_{n_0} \in s^g_{p_{n_0}}$ and $s_n \in s^g_{p_n}$). Then $x_G \cap z \subseteq x_{p_{n_0}} \cap s_{n_0}$ but $x_{p_{n_0}} \cap s_{n_0}$ is finite.

By the second condition we have $[S^g_G] \subseteq [T]$ but our complete Laver tree $T \in V$ is almost disjoint i.e. collection of all branches in T are almost disjoint in the ground model but

$$(\forall x)(\forall y)(\forall n \in \omega)(x \neq y \wedge x \restriction n \in T \wedge y \restriction n \in T) \longrightarrow (\exists m \in \omega)(|x \cap y| < m)$$

is $\prod^1_1$ formula and then is absolute between transitive ZF models of the set theory. Then our tree T consists almost disjoint branches in the generic extension $V[G]$ and then $[S^g_G]$ forms almost disjoint family also. Then $\{x_G\} \cup [S^g_G] \cup (\omega^\omega \cap V)$ forms almost disjoint family.

The similar argument shows that $[S^b_G] \cup (\omega^\omega \cap V)$ forms almost disjoint family. □

CLAIM 2.9. x_G *is dominating in* $\omega^\omega \cap V$.

PROOF. Let us consider any $y \in \omega^\omega \cap V$ then we can find a generic condition $p \in G$ such that $y \in \mathcal{F}_p$. Let $m = dom x_p$ (here $x_p \subseteq x_G$) and for any $n \in \omega$ with $m < n$ then by 3) condition of order the set

$$D_{y,n} = \{p \in Q_T : y(n) < x_p(n)\} \in V$$

is dense in under p because each node of T is ω-splitting one. $\square$

Now let us consider any cardinal κ greater than ω_1 with a uncountable cofinality and finite support iteration $((P_\alpha : \alpha \leq \kappa), (\dot{Q}_\beta : \beta < \kappa))$ such that for every $\beta < \kappa$ we have $\Vdash_{P_\beta} \dot{Q}_\beta = \hat{Q}_T$. Assume that $G_\beta = \{p \in P_\beta : i_{\beta\kappa}(p) \in G\}$ where $G \supset P_\kappa$ generic filter over V and $\beta < \kappa$. Then there exists $H \subseteq \dot{Q}_{\beta G_\beta}$ generic over universe $V[G_\beta]$ such that $G_{\beta+1} = G_\beta * H$. Now let us define the following family $\mathcal{A}_\beta = \{x_{G_{\beta+1}}\} \cup [S^g_{G_{\beta+1}}]$ and then $\mathcal{A} = \bigcup\{\mathcal{A}_\beta : \beta < \kappa\}$. In $V[G]$ we show that $\mathcal{A}$ forms **a.d.** and for every $\mathcal{B}$ **m.a.d.** family containing $\mathcal{A}$. Let us consider any two different reals $x, y \in \mathcal{A}$. Then there are $\alpha, \beta < \kappa$ such that $x \in \mathcal{A}_\alpha$ and $y \in \mathcal{A}_\beta$. We can assume that $\alpha \leq \beta$ (for the other case the proof is the same). First assume that $\alpha < \beta$ then $x \in \omega^\omega \cap V[G_\alpha]$ and if $y = y_{G_{\beta+1}}$ or $y \in [S^g_{G_{\beta+1}}]$ then by the Claim 2.8 we have that $x \cap y$ is finite. If $\alpha = \beta$ then we can assume that $x = x_{G_{\beta+1}}$ and $y \in [S^g_{G_{\beta+1}}]$ and once again by the Claim 2.8 the intersection $x \cap y$ is finite too.

Now let us choose in $V[G]$ any complete Laver tree $T_1 \subseteq T$ which is a subtree of the tree T. Then by choosing a nice name for T_1 there is a some $\beta < \kappa$ such that $T_1 \in V[G_\beta]$. Then by the Claim 2.6 there is a some real $z \in [S^b_{G_{\beta+1}}] \subseteq T$ such that $z \in T_1$ and $z \cap x_{G_{\beta+1}}$ is infinite. Moreover, let observe that $z \notin \mathcal{B}$ because in other case $x_{G_{\beta+1}}, z \in \mathcal{B}$ what witness that $\mathcal{B}$ is not an a.d. family, contradiction. By the Claim 2.7 we have $[S^g_{G_{\beta+1}}] \cap [T_1] \neq \emptyset$. Then we have showed that $\mathcal{B}$ is a cl-nonmeasurable set in the generic extension $V[G]$. $\square$

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